# Turning lax monoidal categories into strict ones

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# Plan of the presentation

1. Colax monoidal categories & lax monoidal functors
2. Strictification process
3. Bonus features, generalizations

# Colax monoidal categories

## Definition

A *colax monoidal category* $(\mathcal{A}, \otimes, \gamma, \iota)$ is a category $\mathcal{A}$ together with:

- a functor $\otimes_n : \mathcal{A}^n \to \mathcal{A}$ for all $n \geqslant 0$, denote $a_1 \otimes \cdots \otimes a_n := \otimes_n(a_1, \ldots, a_n)$, $[a] := \otimes_1(a)$, $I := \otimes_0(*)$
- for every list of lists $(\vec{a_1}, \ldots, \vec{a_k})$ of objects of $\mathcal{A}$ an *associator* morphism. For instance:

$$\gamma_{((a_1,a_2),(a_3,a_4))} : a_1 \otimes a_2 \otimes a_3 \otimes a_4 \to (a_1 \otimes a_2) \otimes (a_3 \otimes a_4),$$
$$\gamma_{((a_1),(),(a_2,a_3))} : a_1 \otimes a_2 \otimes a_3 \to [a_1] \otimes I \otimes (a_2 \otimes a_3).$$

- for every object $a \in \mathcal{A}$ a *unitor* morphism:

$$\iota_a : [a] \to a.$$

These are subject to *associativity* and *unit* laws.

## Remark - variants

- $\gamma, \iota$ identities $\Rightarrow$ a *strict monoidal category*,
- $\gamma, \iota$ isomorphisms $\Rightarrow$ an *unbiased monoidal category*. (are equivalent to ordinary monoidal categories [Leinster, 2004, I.3]).
- $\iota$ is the identity $\Rightarrow$ a *normal* colax monoidal category.

## Example, [BW, 2011]

Let $(t, \delta, \epsilon)$ be a comonad on a category $\mathcal{A}$ with coproducts. Define a colax monoidal category structure on $\mathcal{A}$ by putting:

$$a_1 \otimes \cdots \otimes a_n := \coprod_{i=1}^{n} ta_i.$$

The unitor is given by the comonad counit: $\iota_a := \epsilon_a : [a] = ta \to a$. The associator is given by:

$$\begin{array}{rcl} a_1 \otimes a_2 \otimes b = ta_1 + ta_2 + tb & \xrightarrow{\delta_{a_1} + \delta_{a_2} + \delta_{a_3}} & t^2 a_1 + t^2 a_2 + t^2 b \\ & & \downarrow \cong \\ (a_1 \otimes a_2) \otimes [b] \otimes I = t(ta_1 + ta_2) + t^2 b + t\varnothing & \longleftarrow & t^2 a_1 + t^2 a_2 + t^2 b + \varnothing \end{array}$$

Recall that a *multicategory* $\mathcal{M}$ consists of:

- a set of objects ob $\mathcal{M}$,
- for every $(n+1)$-tuple of objects $(a_1, \dots, a_n, b)$ a set $\mathcal{M}(a_1, \dots, a_n; b)$. We denote $f \in \mathcal{M}(a_1, \dots, a_n; b)$ by:

$$f : a_1, \dots, a_n \to b,$$

- an identity map $1_a \in \mathcal{M}(a; a)$ for all objects $a$,
- suitable composition operation,

subject to associativity and unit laws.

**Example**

$\mathrm{Vect}_k$. A morphism $f : V_1, \dots, V_n \to W$ is a $k$-multilinear map $V_1 \times \cdots \times V_n \to W$.

# Multicategories vs monoidal categories

## Proposition

There is a functor $I : \mathrm{ColaxMonCat}_I \to \mathrm{Mult}$ sending a colax monoidal category $(\mathcal{A}, \otimes, \gamma, \iota)$ to its *underlying multicategory*. Its objects are the objects of $\mathcal{A}$, its hom set is given by:

$$I\mathcal{A}(a_1, \ldots, a_n; b) := \mathcal{A}(a_1 \otimes \cdots \otimes a_n, b).$$

A multicategory lies in the essential image of $I$ if and only if it is *weakly representable*.

# Lax monoidal functors

## Definition

A *lax monoidal functor* $(F, \overline{F}) : (\mathcal{A}, \otimes, \gamma, \iota) \to (\mathcal{B}, \odot, \gamma', \iota')$ between colax monoidal categories consists of:

- a functor $F : \mathcal{A} \to \mathcal{B}$,
- for every $(a_1, \ldots, a_n)$ a morphism in $\mathcal{B}$:
  $\overline{F}_{a_1,\ldots,a_n} : Fa_1 \odot \cdots \odot Fa_n \to F(a_1 \otimes \cdots \otimes a_n)$,

subject to associativity and unit axioms.

## Example

Take two sup-semilattices $A, B$ with a lowest element. Regard them as monoidal categories $(A, \vee)$, $(B, \vee)$.
**Any** order-preserving map $f : A \to B$ automatically lax monoidal because we have:

$$f(a) \vee f(b) \leqslant f(a \vee b).$$

## Example

A lax monoidal functor $* \to (\mathcal{A}, \otimes, \gamma, \iota)$ is precisely a **monoid** in the monoidal category $\mathcal{A}$.

## Example

The forgetful functor $U : \mathrm{Vect}_k \to \mathrm{Set}$ becomes lax monoidal if we define:

$$\overline{U}_{V_1,\ldots,V_n} : UV_1 \times \cdots \times UV_n \to U(V_1 \otimes_k \cdots \otimes_k V_n),$$
$$(v_1, \ldots, v_n) \mapsto v_1 \otimes \cdots \otimes v_n.$$

# Turning colax monoidal categories into strict ones

**Question**: How to turn a colax monoidal category into a strict one in a best possible way?

**Answer**: By applying the left 2-adjoint:

$$\mathrm{StrMonCat}_s \underset{\hookrightarrow}{\overset{\dashleftarrow}{\perp}} \mathrm{ColaxMonCat}_l$$

$\mathrm{StrMonCat}_s$ is 2-cocomplete, so by [Lack, 2002], this 2-adjoint exists. **What does it look like?**

# Ingredient: partial evaluations

## Definition [PF2020]

Let $(\mathcal{A}, \otimes, \gamma, \iota)$ be a colax monoidal category. A *partial evaluation* $(\vec{a_1}, \dots, \vec{a_k})$ is a list of lists of objects of $\mathcal{A}$.

Its *source* and *target* are defined to be $\mathrm{conc}(\vec{a_1}, \dots, \vec{a_k})$ and $(\otimes_{n_1}(\vec{a_1}), \dots, \otimes_{n_k}(\vec{a_k}))$.

## Example

For $(\mathbb{N}, +)$, an example is:

$$((1, 2), (4, 1, 0), (7)) : (1, 2, 4, 1, 0, 7) \to (3, 5, 7),$$
$$((), (), (1)) : (1) \to (0, 0, 1).$$

## Example

For a terminal monoidal category $*$, a partial evaluation is precisely an *order-preserving functions* between finite ordinals. For instance:

$$((\bullet, \bullet), (), (\bullet), ()) : (\bullet, \bullet, \bullet) \to (\bullet, \bullet, \bullet, \bullet)$$

corresponds to the unique order-preserving function $f : \{0 \to 1 \to 2\} \to \{0 \to 1 \to 2 \to 3\}$ whose fibres over $0, 1, 2, 3$ have sizes $2, 0, 1, 0$.

# Category of corners

**Construction**: Let $(\mathcal{A}, \otimes, \gamma, \iota)$ be a colax monoidal category. Define its *category of corners* $\mathrm{Cnr}(\mathcal{A})$ as follows.

- the objects $(a_1, \ldots, a_n)$ are lists of objects of $\mathcal{A}$,
- a morphism has two components: partial evaluation, list of morphisms of $\mathcal{A}$. For example:

$$
\begin{array}{ccc}
(a_1, a_2, a_3) & \xrightarrow{((),(a_1,a_2),(),(a_3))} & (I, a_1 \otimes a_2, I, [a_3]) \\
 & & \big\downarrow{\scriptstyle (f_1,f_2,f_3,f_4)} \\
 & & (b_1, b_2, b_3, b_4)
\end{array}
$$

- the identity morphism is defined as the following corner:

$$\begin{array}{ccc} (a_1, \dots, a_n) & \longrightarrow & ([a_1], \dots, [a_n]) \\ & & \Big\downarrow {\scriptstyle (\iota_{a_1}, \dots, \iota_{a_n})} \\ & & (a_1, \dots, a_n) \end{array}$$

# Category of corners - composition (1 of 6)

An example of composition of two corners:

$$(a_1, a_2, a_3) \xrightarrow{((a_1,a_2),(),(a_3))} (a_1 \otimes a_2, I, [a_3])$$

$$(a_1 \otimes a_2, I, [a_3]) \xrightarrow{(f_1,f_2,f_3)} (b_1, b_2, b_3)$$

$$(b_1, b_2, b_3) \xrightarrow[((b_1),(b_2,b_3))]{} ([b_1], b_2 \otimes b_3)$$

$$([b_1], b_2 \otimes b_3) \xrightarrow{(g_1,g_2)} (c_1, c_2)$$

# Category of corners - composition (2 of 6)

$$
\begin{array}{ccccc}
(a_1, a_2, a_3) & \xrightarrow{((a_1,a_2),(),(a_3))} & (a_1 \otimes a_2, I, [a_3]) & \overset{((a_1 \otimes a_2),(I,[a_3]))}{\dashrightarrow} & ([a_1 \otimes a_2], I \otimes [a_3]) \\
 & & \Big\downarrow {\scriptstyle (f_1,f_2,f_3)} & & \Big\downarrow {\scriptstyle ([f_1],f_2 \otimes f_3)} \\
 & & (b_1, b_2, b_3) & \xrightarrow[((b_1),(b_2,b_3))]{} & ([b_1], b_2 \otimes b_3) \\
 & & & & \Big\downarrow {\scriptstyle (g_1,g_2)} \\
 & & & & (c_1, c_2)
\end{array}
$$

# Category of corners - composition (3 of 6)

$$(a_1, a_2, a_3) \xrightarrow{((a_1,a_2),(),(a_3))} (a_1 \otimes a_2, I, [a_3]) \xrightarrow{((a_1 \otimes a_2),(I,[a_3]))} ([a_1 \otimes a_2], I \otimes [a_3])$$

$$([a_1 \otimes a_2], I \otimes [a_3]) \xrightarrow{(g_1 \circ [f_1], g_2 \circ f_2 \otimes f_3)} (c_1, c_2)$$

# Category of corners - composition (4 of 6)

$$(a_1, a_2, a_3) \overset{((a_1,a_2),(a_3))}{\dashrightarrow} (a_1 \otimes a_2, [a_3])$$

$$(a_1, a_2, a_3) \xrightarrow{((a_1,a_2),(),(a_3))} (a_1 \otimes a_2, I, [a_3]) \xrightarrow{((a_1 \otimes a_2),(I,[a_3]))} ([a_1 \otimes a_2], I \otimes [a_3])$$

$$([a_1 \otimes a_2], I \otimes [a_3]) \xrightarrow{(g_1 \circ [f_1], g_2 \circ f_2 \otimes f_3)} (c_1, c_2)$$

# Category of corners - composition (5 of 6)

$$(a_1, a_2, a_3) \xrightarrow{((a_1,a_2),(a_3))} (a_1 \otimes a_2, [a_3])$$

$$(a_1 \otimes a_2, [a_3]) \xrightarrow{(\gamma,\gamma)} ([a_1 \otimes a_2], I \otimes [a_3])$$

$$(a_1, a_2, a_3) \xrightarrow[((a_1,a_2),(),(a_3))]{} (a_1 \otimes a_2, I, [a_3]) \xrightarrow[((a_1 \otimes a_2),(I,[a_3]))]{} ([a_1 \otimes a_2], I \otimes [a_3])$$

$$([a_1 \otimes a_2], I \otimes [a_3]) \xrightarrow{(g_1 \circ [f_1], g_2 \circ f_2 \otimes f_3)} (c_1, c_2)$$

# Category of corners - composition (6 of 6)

$$\begin{array}{ccc} (a_1, a_2, a_3) & \xrightarrow{((a_1,a_2),(a_3))} & (a_1 \otimes a_2, [a_3]) \\ & & \Big\downarrow {\scriptstyle (g_1 \circ [f_1] \circ \gamma, g_2 \circ f_2 \otimes f_3 \circ \gamma)} \\ & & (c_1, c_2) \end{array}$$

# Category of corners

The tensor product $\boxplus$ in $\mathrm{Cnr}(\mathcal{A})$ is given by concatenation:

$$(a_1, \ldots, a_n) \boxplus (b_1, \ldots, b_m) := (a_1, \ldots, a_n, b_1, \ldots, b_m),$$

The unit element is given by the empty list $I := ()$.

There is a lax monoidal functor $(P, \overline{P}) : (\mathcal{A}, \otimes, \gamma, \iota) \to (\mathrm{Cnr}(\mathcal{A}), \boxplus, I)$:

$$\begin{array}{ccccc} a & & & (a) & \longrightarrow & ([a]) \\ & & & & & \downarrow{\scriptstyle (\iota_a)} \\ {\scriptstyle f}\downarrow & & \mapsto & & & (a) \\ & & & & & \downarrow{\scriptstyle (f)} \\ b & & & & & (b) \end{array}$$

The lax monoidal structure, a collection of 1-cells in $\mathrm{Cnr}(\mathcal{A})$ like this:

$$\overline{P}_{a_1,\ldots,a_n} : Pa_1 \boxplus \cdots \boxplus Pa_n \to P(a_1 \otimes \cdots \otimes a_n),$$

is given by the corner:

$$\begin{array}{ccc} (a_1, \ldots, a_n) & \longrightarrow & (a_1 \otimes \cdots \otimes a_n) \\ & & \| \\ & & (a_1 \otimes \cdots \otimes a_n) \end{array}$$

## Theorem

The category of corners construction is left adjoint to the inclusion:

$$\mathrm{StrMonCat}_s \;\; \overset{\mathrm{Cnr}(-)}{\underset{\hookrightarrow}{\overset{\longleftarrow}{\perp}}} \;\; \mathrm{ColaxMonCat}_l$$

In other words:

$$\begin{array}{ccc} & & (\mathcal{B},\odot) \\ & \overset{\forall \text{ lax monoidal}}{\nearrow} & \uparrow {\scriptstyle \exists! \text{ strict monoidal}} \\ (\mathcal{A},\otimes,\gamma,\iota) & \xrightarrow[(P,\overline{P})]{} & (\mathrm{Cnr}(\mathcal{A}),\boxplus) \end{array}$$

## Sketch of a proof.

For a lax monoidal functor $(F, \overline{F}) : (\mathcal{A}, \otimes, \gamma, \iota) \to (\mathcal{B}, \odot)$, the unique strict monoidal functor $F' : (\mathrm{Cnr}(\mathcal{A}), \boxplus) \to (\mathcal{B}, \odot)$ sends
$(a_1, \dots, a_n) \mapsto Fa_1 \odot \cdots \odot Fa_n$,

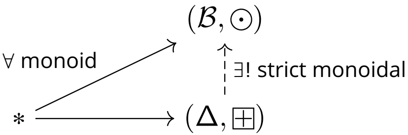
and on morphisms it is defined like this:

$$
\begin{array}{ccc}
(a_1, a_2, a_3) \xrightarrow{((a_1,a_2),(),(a_3))} (a_1 \otimes a_2, I, [a_3]) & & Fa_1 \odot Fa_2 \odot Fa_3 \\
 & & \| \\
 & & Fa_1 \odot Fa_2 \odot I \odot Fa_3 \\
\Big\downarrow (f_1,f_2,f_3) & \mapsto & \Big\downarrow \overline{F}_{a_1,a_2} \odot \overline{F}_0 \odot \overline{F}_{a_3} \\
 & & F(a_1 \otimes a_2) \odot FI \odot F[a_3] \\
 & & \Big\downarrow Ff_1 \odot Ff_2 \odot Ff_3 \\
(b_1, b_2, b_3) & & Fb_1 \odot Fb_2 \odot Fb_3
\end{array}
$$

□

## Example

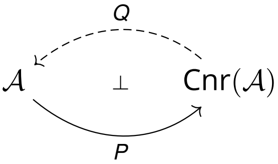
In particular, Cnr$(*) = \Delta$, the category of finite ordinals and order-preserving maps. It enjoys the universal property that:



# Lax coherence theorem

**Theorem**

The underlying functor $P : \mathcal{A} \to \mathrm{Cnr}(\mathcal{A})$ of the unit of the above adjunction admits a left adjoint whose counit is given by the unitor of $(\mathcal{A}, \otimes, \gamma, \iota)$:




In particular, every **normal** colax monoidal category can be reflectively embedded in a strict monoidal category.

## Sketch of a proof.

The left adjoint $Q : \mathrm{Cnr}(\mathcal{A}) \to \mathcal{A}$ sends $(a_1, \ldots, a_n) \mapsto a_1 \otimes \cdots \otimes a_n$, and on morphisms is defined like this:

$$
\begin{array}{ccccc}
(a_1, a_2, a_3) \xrightarrow{((a_1,a_2),(a_3))} (a_1 \otimes a_2, [a_3]) & & & & a_1 \otimes a_2 \otimes a_3 \\
 & & & & \downarrow \gamma \\
\downarrow (f_1, f_2) & & \mapsto & & (a_1 \otimes a_2) \otimes [a_3] \\
 & & & & \downarrow f_1 \otimes f_2 \\
(c_1, c_2) & & & & c_1 \otimes c_2
\end{array}
$$

$\square$

# Bonus features

# Relationship to the Kleisli category (1 of 2)

## Remark

Every colax monoidal category $(\mathcal{A}, \otimes, \gamma, \iota)$ has an underlying comonad $(\mathcal{A}, [-] : \mathcal{A} \to \mathcal{A}, \gamma_{((-))}, \iota)$, with the comultiplication $[a] \to [[a]]$ being given by the associator $\gamma$ evaluated at $((a))$.

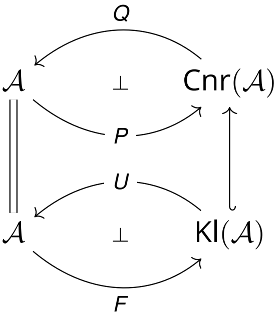
There is an obvious functor $\mathsf{Kl}(\mathcal{A}) \to \mathsf{Cnr}(\mathcal{A})$:

$$\begin{array}{ccc} a & & (a) \longrightarrow ([a]) \\ f \big\downarrow\!\!\wr & \mapsto & \big\downarrow (f) \\ b & & (b) \end{array}$$

# Relationship to the Kleisli category (2 of 2)

## Proposition

The adjunction from the above theorem restricts to the Kleisli adjunction:



# Relationship to multicategories

Recall the adjunction between strict monoidal categories and multicategories:

$$\mathsf{StrMonCat}_s \xrightarrow{J} \mathsf{ColaxMonCat}_l \xrightarrow{I} \mathsf{Multicat}, \qquad F : \mathsf{Multicat} \to \mathsf{StrMonCat}_s, \quad F \dashv IJ$$

Since $I$ is fully faithful, this induces an adjunction:

$$\mathsf{StrMonCat}_s \underset{J}{\overset{FI}{\leftrightarrows}} \mathsf{ColaxMonCat}_l, \qquad FI \dashv J$$

As adjoints are unique up to an isomorphism, we have:

$$\mathsf{Cnr}(\mathcal{A}) \cong FI\mathcal{A}.$$

# Some bonus observations

- Given $(\mathcal{A}, \otimes, \gamma, \iota)$ with $\gamma, \iota$ invertible, a strict monoidal category $\mathcal{A}'$ **equivalent** to $\mathcal{A}$ can be obtained from Cnr$(\mathcal{A})$ by formally inverting partial evaluations,
- The strict monoidal category (Cnr$(\mathcal{A}), \boxplus)$ is *flexible*: any monoidal functor out of Cnr$(\mathcal{A})$ is naturally isomorphic to a strict one
- The construction can be generalized to symmetric and braided monoidal variants - Cnr$(\mathcal{A})$ can be described in terms of generators and relations.

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Thank you.